Leak Rate Problem

Using a specification from the Parenteral Society, the leak rate for a new, clean empty freeze dryer should be less than 0.02mBar-L/sec. a) If one had a leak of that magnitude, and if it came from multiple sources, where each source was a round hole with diameter 0.2 microns, then how many such holes would be present? b) If the leak rate were changed to X mBar-L/sec, then how many such holes would be present?

What follows will be the derivation of an equation that will allow us to solve the problem above as well as many similar problems. The equation that will be derived is

$\text{Area} = R_L \cdot \frac{\sqrt{2 \cdot \pi \cdot R \cdot T \cdot M}}{P \cdot R \cdot T}$ where R_L is the "Leak Rate" in mbar-Liters/sec or PV / time.

M is molecular weight of air, P is pressure outside the LYO (or more technically, it is ΔP between the inside and outside of the Lyo), R is the Molar Gas constant, $8.314472 \cdot \frac{\text{joule}}{\text{mole} \cdot \text{K}}$ and T is room temperature in degrees K. [DO ALL CALCULATIONS IN SI UNITS ALL THE TIME AND THEN CONVERT THE FINAL ANSWER TO WHATEVER UNITS YOU WANT.]

I use MathCad. I can multiply $(2\text{ft} \cdot 3\text{mm} \cdot 0.05\text{furlong}) = 18.395\,\text{L}$ and MathCad assures that my answer and units are correct. If you aren't using MathCad, Be Careful.

Kinetic theory defines ν as the number of $\frac{\text{collisions}}{\text{m}^2 \cdot \text{sec}}$ that will occur between a gas some area of a wall.

$\nu = \frac{n' \cdot \nu_m}{4}$ EQ #1. n' is defined as the number of molecules per unit volume.

$n' = \frac{\text{molecules}}{\text{m}^3}$ and ν_m is the mean velocity of gas molecules from kinetic theory

$$\nu_m = \sqrt{\frac{8 \cdot R \cdot T}{\pi \cdot M}}$$

Thus we began with 1 equation and 2 definitions of terms.

$\nu = \frac{n' \cdot \nu_m}{4}$ EQ #1 Let ν = the number of collisions per square meter in a sec.

$n' = \frac{molecules}{m^3}$ Definition n' is a concentration 'like' term (conc. is $\frac{moles}{m^3}$)

$\nu_m = \sqrt{\frac{8 \cdot R \cdot T}{\pi \cdot M}}$ Defiition Note: $R := 8.314472 \frac{J}{mol \cdot K}$ $M := 0.029 \frac{kg}{mol}$ = air molecular wt.

and $T := 298.15K$ = room temperature.

Since we know that the collisions are done with air, we can estimate the mass, w, of the collisions.

$w = \frac{\nu \cdot M}{N_a}$ EQ#2 w = total mass of collisions per m^2 in a second.

M = mass of each air molecule

Na = Avogodro's number $N_a := 6.0221415 \cdot 10^{23} \cdot mole^{-1}$

as a check on what has been done so far, we can look at a units only calculation:

$$w = \frac{\left(\frac{collisions}{m^2 \cdot s}\right) \cdot .029 \frac{kg}{mol}}{6.0221415 \cdot 10^{23} \cdot mole^{-1}} = \blacksquare \frac{kg}{m^2 \cdot s}$$

Substituting ν from EQ#1 into EQ#2 and solving for the number of molecules/m^3

$\nu = \frac{n' \cdot \nu_m}{4}$ EQ#1 $w = \frac{\nu \cdot M}{N_a}$ EQ#2

$$w = \frac{\frac{n' \cdot \nu_m}{4} \cdot M}{N_a}$$ <= This is the substitution

$$n' = \frac{4 \cdot w \cdot N_a}{\nu_m \cdot M}$$ EQ#3 <= Here it is solved for the concentration 'like' term.

Now verify the units for n', the concentration 'like' term: $\frac{n'}{N_a} = \frac{\frac{molecules}{m^3}}{\frac{molecules}{mole}} = \frac{mol}{m^3}$

note that n'/Na is an actual concentration term and can be substituted into the gas law.

Gas Law: $P = \frac{n}{V} \cdot R \cdot T$ please see that $\frac{n}{V}$ is the same as $\frac{n'}{Na}$. Both are concentration.

$$P = \frac{n'}{N_a} \cdot R \cdot T = \frac{4 \cdot w}{v_m \cdot M} \cdot R \cdot T$$

EQ#4 Notice that for n' we substituted in EQ#3 and cancelled out N_a

$$P = \frac{4w}{v_m \cdot M} \cdot R \cdot T \qquad \text{EQ\#4}$$

Now substitute in an expression for the mean velocity of molecules, $\mathbf{v_m}$

$$P = \frac{4 \cdot w \cdot R \cdot T}{\sqrt{\frac{8 \cdot R \cdot T}{\pi \cdot M}} \cdot M}$$ and simplify to get $$P = w \cdot \sqrt{\frac{2 \cdot \pi \cdot R \cdot T}{M}} \qquad \text{EQ\#5}$$

'w' is the mass of molecules/sec striking some area and is easily obtained. From the problem, we know that 0.02 mbar-liter /s of molecules are striking an unknown area. And we can safely assume that they are air molecules with M = 29 g/mol. The mass of the molecules was given by the problem, where the Leak Rate was defined as mbar-Liter/sec.

Writing a differential form of the gas law,

$$\frac{d}{dt}n = \frac{\left(\frac{d}{dt}P\right) \cdot V}{R \cdot T}$$ we should note that $\left(\frac{d}{dt}P\right) \cdot V$ is in fact our leak rate. We should call it

$$R_L = \left(\frac{d}{dt}P\right) \cdot V$$

but notice that $\frac{d}{dt}n \cdot \frac{M}{m^2} = w = \frac{\frac{d}{dt}P \cdot V \cdot M}{R \cdot T \cdot Area}$ Not obvious? w has units of $\frac{kg}{m^2 \cdot s}$

So substitute in R_L and we get

$$w = \frac{R_L \cdot M}{R \cdot T \cdot Area}$$ which can now be put back into EQ#5 to get

$$P = \frac{R_L}{R \cdot T \cdot Area} \cdot M \cdot \sqrt{\frac{2 \cdot \pi \cdot R \cdot T}{M}}$$ which simplifies to $$Area = R_L \cdot \frac{\sqrt{2 \cdot \pi \cdot R \cdot T \cdot M}}{P \cdot R \cdot T}$$

Now let's put in some numbers so that we can compute. $mbar := .001bar$ <= definition

$R_L := 0.02 \cdot \frac{mbar \cdot liter}{s}$ Leak Rate

$R := 8.314472 \frac{J}{mol \cdot K}$ Molar Gas Constant

$P := 1.013 \times 10^5 \, Pa - 13.332Pa$ Pressure differential between atmospheric and Lyo

$T := 298.16K$

$M := 29 \frac{gm}{mol}$ Molecular weight of Air

$Area := R_L \cdot \frac{\sqrt{2 \cdot \pi \cdot R \cdot T \cdot M}}{P \cdot R \cdot T}$ $Area = 1.693 \times 10^{-10} m^2$ $dia := 2 \cdot \sqrt{\frac{Area}{\pi}}$

$dia = 14.681 \, \mu m$ <= diameter if it were a single hole

The original question was how many 0.2µm holes must exist to obtain the observed leak rate?

Area of a .2µm hole is $BugArea := \left(\frac{.2\mu m}{2}\right)^2 \cdot \pi$ $BugArea = 0.031 \, \mu m^2$

$\frac{Area}{BugArea} = 5389$ That many pores of 0.2µm diameter can exist with the Society Leak Rate.